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METHODS FOR DETERMINING FREQUENCY RESPONSE OF ENGINES AND CONTROL SYSTEMS FROM TRANSIENT DATA

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SUMMARY

Methods are presented that use general correlative time-response input and output data for a linear system to determine the frequency-response function of that system. These methods give an exact description of any linear system for which such transient data are available.

Examples are shown of application of a method to both an underdamped and a critically damped exact second-order system, and to an exact first-order system with and without dead time. Experimental data for a turbine-propeller engine showing the response of engine speed to change in propeller-blade angle are presented and analyzed.

INTRODUCTION

A basic problem confronting the designer of engine controls is that of determining the behavior of the complete engine and control system under varying conditions of operation. This question becomes particularly acute when various engine parameters, for example, shaft torque or turbine-inlet temperature, must be closely controlled in order to prevent engine damage or even failure. The problem then becomes one of matching transient behavior of the control to that of the engine in order to attain a desired system response.

General methods do not exist for the solution of the equations of motion of nonlinear systems. As a result, the analysis of such systems may be impracticably difficult, if, indeed, a solution can be found at all. Because the behavior of many nonlinear systems may be satisfactorily approximated by the assumption of system linearity and because general mathematical methods and techniques for the analysis of linear systems are readily available and (in comparison with present nonlinear methods) relatively simple to apply, the assumption is generally made that the

system being studied is linear. The methods of this report are based on such an assumption.

Approaches utilized in the analysis of linear systems (reference 1, pp. 17-18) are: (1) transient analysis, in which the characteristic time response of the system is determined for standard inputs such as step and impulse functions, or (2) frequency analysis, in which system behavior is characterized by the steady-state amplitude and phase relations of the system input and output for sinusoidal inputs of various frequencies.

For inputs such as step or impulse functions, the inherent system characteristics might be obtained by fitting an equation to the output function and from it deriving the differential equation of the system. If the input and output functions were of any general form, fitting differential equations to the data might still be possible, but for systems of inherently high order the accuracy of such a procedure would be low. In general, use of the time-response form of system description involves dealing with convolution integrals (reference 2, p. 54) when the unit so described is to be combined with other units. For complex systems, this descriptive form is not readily adapted to manipulation.

A linear system is known to be characterized by its steady-state response to all frequencies of sinusoidal inputs. The frequency-response form is useful for the description of linear systems because of the ease with which various system characteristics can be manipulated in dealing with combinations of units. For example, the over-all amplitude ratio of output to input for any frequency of input to a system consisting of several units in series is obtained by a simple multiplication of the amplitude ratios for the several units. A possible difficulty in the application of this method to analysis of engines is the necessity for maintaining a sinusoidal input of constant amplitude and frequency for a length of time sufficient to insure disappearance of transient effects in the output.

The nature of real physical systems may make actual imposition or measurement of step or impulse inputs impossible; in addition, sinusoidal inputs may prove impracticable. The need thus arises for feasible methods capable of handling data of any general form. Such methods that use general correlative time-response data for system input and output to determine the frequency-response characteristics of the system have been developed at the NACA Lewis laboratory and are presented in this report. These methods give an exact description of any linear system for which such data are available.

Three exact methods of obtaining the frequency-response function of a system are shown for system-equilibrium final conditions. A graphical approximation to one of the methods is also given. Modification of one of the exact methods to account for oscillatory final conditions is shown and treatment of dead time is presented. As an illustration of one of the methods, three examples are given, two based on analytically determined system-response curves and one on experimental data obtained at this laboratory for a turbine-propeller engine.

SYMBOLS

The following symbols are used in this report:

A	amplitude of system steady oscillation
a,b	constants
$F(i\omega)$	system frequency-response function
$F(p)$	system transfer function
$f(t)$	general function of time
$f'(t)$	first time derivative of $f(t)$
$G(p)$	system transfer function including dead time
L	Laplace transform
p	complex number
R	amplitude ratio of $F(i\omega)$
t	time, seconds
x	general time-dependent input
y	general time-dependent output
$y^n(t)$	n^{th} time derivative of y
β	frequency of system steady oscillation, radians per second
Δt	system dead time, seconds
ζ	damping ratio

θ phase angle of $F(i\omega)$, radians

τ system time constant, seconds

ϕ phase angle of system steady oscillation, radians

ω frequency of system sinusoidal input, radians per second

Subscripts:

0 last term of differential equation

f final value

k general term of series

m degree of differential equation of input

n degree of differential equation of output

r last term of series

ANALYSIS

The system considered in the following derivation is assumed to be linear. In particular, behavior of the system is assumed representable by a linear differential equation with constant coefficients. The block diagram of a system having an input x and a corresponding output y is shown in figure 1. The symbol $F(p)$ inside the box represents the system operator which, acting on the input x, yields the output y. The linear differential equation relating x and y may be written as

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_0 x \tag{1}$$

If the system is initially at rest (or in equilibrium), termwise application to equation (1) of the Laplace transformation (reference 2, pp. 51 to 56), defined as

$$L\left[f(t)\right] = \int_0^\infty f(t) e^{-pt} \, dt \tag{2}$$

and factoring of the result give

$$(a_n p^n + a_{n-1} p^{n-1} + \ldots + a_0)\, L(y) = (b_m p^m + b_{m-1} p^{m-1} + \ldots + b_0) L(x)$$

or

$$L(y) = \frac{b_m p^m + b_{m-1} p^{m-1} + \ldots + b_0}{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_0} L(x) \tag{3}$$

Equation (3) may be formally written as

$$L(y) = F(p)\, L(x) \tag{4}$$

where

$$F(p) = \frac{b_m p^m + b_{m-1} p^{m-1} + \ldots + b_0}{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_0}$$

The transfer function of the system $F(p)$ is defined as the ratio of the Laplace transform of any normal response of the system to the Laplace transform of the input producing that response. Normal response is the response of the system when initially at rest (reference 2, p. 26).

From equations (2) and (4), the system transfer function may be written as

$$F(p) = \frac{L(y)}{L(x)}$$

or

$$F(p) = \frac{\int_0^\infty y e^{-pt}\, dt}{\int_0^\infty x e^{-pt}\, dt} \tag{5}$$

Equilibrium Final Conditions

Derivation of frequency-response function. - The frequency-response function $F(i\omega)$ is formally obtained directly from equation (5) by replacing p by $i\omega$ (reference 1, pp. 96-98). Then, for a sinusoidal input x of frequency ω, the output y

ultimately is sinusoidal at the same frequency but with a relative magnitude and phase angle equal to the magnitude and the phase angle of the complex number $F(i\omega)$. The frequency-response function is defined as

$$F(i\omega) = \frac{\int_0^\infty ye^{-i\omega t}\,dt}{\int_0^\infty xe^{-i\omega t}\,dt} \tag{6}$$

The term $e^{-i\omega t}$ is oscillatory and the integrals of equation (6) may not converge unless $\int_0^\infty |y|dt$ and $\int_0^\infty |x|dt$ converge. These conditions impose a severe restriction on the types of function that may be used in equation (6) because of the implied requirement that x and y vanish as t increases without limit.

As will be shown, the previous restriction on x and y may be removed by suitable modification of equation (6). A much wider selection of input and output functions is thus permitted.

For a system initially at rest,

$$L\left[f'(t)\right] = pL\left[f(t)\right] \tag{7}$$

An alternate expression for the transfer function then is

$$F(p) = \frac{L(y')}{L(x')} \tag{8}$$

or, from equations (2) and (8),

$$F(p) = \frac{\int_0^\infty y'e^{-pt}\,dt}{\int_0^\infty x'e^{-pt}\,dt}$$

The frequency-response function then becomes

$$F(i\omega) = \frac{\int_0^\infty y' e^{-i\omega t}\, dt}{\int_0^\infty x' e^{-i\omega t}\, dt} \tag{9}$$

when p is replaced by $i\omega$.

For any input or output function ending in equilibrium, x' and y' become and remain zero for t equal to or greater than some time t_f. The integrals of equation (9) therefore converge.

From Euler's relation,

$$e^{-i\omega t} = \cos \omega t - i \sin \omega t$$

equation (9) may be written

$$F(i\omega) = \frac{\int_0^\infty y' \cos \omega t\, dt - i \int_0^\infty y' \sin \omega t\, dt}{\int_0^\infty x' \cos \omega t\, dt - i \int_0^\infty x' \sin \omega t\, dt} \tag{10}$$

This form of the frequency-response function can be used with any data for which the input and the output begin and end in a steady state.

The integrals of equation (10) can be evaluated in various ways. One theoretically exact method, which has been used at the NACA Lewis laboratory, utilizes a rolling-sphere harmonic analyzer. This type of analyzer, the same in operating principle as the device described in reference 3, was designed to obtain the Fourier coefficients from cyclic data, but the manner in which the coefficients are determined results in evaluation of integrals of exactly the form of those in equation (10). The frequency-response function can thus be obtained for any frequency by operating the analyzer over the output curve to evaluate the integrals in the numerator and over the input curve to determine the integrals in the denominator of equation (10) without replotting the data in any other form.

The frequency spectra for nonperiodic phenomena are determined in reference 4 by evaluating Fourier integrals with a rolling-sphere

harmonic analyzer in a manner similar to that followed in this report in connection with equation (10). The method of reference 4, however, differs from that of this report in being restricted to functions that begin and end at zero. In addition, the concepts of a system transfer or a system frequency-response function are untreated.

Modifications of frequency-response derivation. - The frequency response may, at times, be desired directly in terms of the input and output functions themselves rather than their time derivatives. For example, if an electronic device were used for analysis, the avoidance of the added complexity and attendant inaccuracy of differentiating circuits might be desirable.

If both x and y come to a steady state by or before some time t_f, equation (9) may be rewritten as

$$F(i\omega) = \frac{\int_0^{t_f} y' e^{-i\omega t}\, dt}{\int_0^{t_f} x' e^{-i\omega t}\, dt} \tag{11}$$

because for $t \geq t_f$ the integrands of equation (11) are zero. Integrating equation (11) by parts gives

$$F(i\omega) = \frac{i\omega \int_0^{t_f} y e^{-i\omega t}\, dt + y e^{-i\omega t}\Big|_0^{t_f}}{i\omega \int_0^{t_f} x e^{-i\omega t}\, dt + x e^{-i\omega t}\Big|_0^{t_f}}$$

or

$$F(i\omega) = \frac{\left(\int_0^{t_f} y \cos \omega t\, dt - y_f \frac{\sin \omega t_f}{\omega}\right) - i\left(\int_0^{t_f} y \sin \omega t\, dt - y_f \frac{\cos \omega t_f}{\omega}\right)}{\left(\int_0^{t_f} x \cos \omega t\, dt - x_f \frac{\sin \omega t_f}{\omega}\right) - i\left(\int_0^{t_f} x \sin \omega t\, dt - x_f \frac{\cos \omega t_f}{\omega}\right)} \tag{12}$$

An alternate procedure, leading to a somewhat simpler expression, employs the principle of superposition. For example, y may be considered the linear combination of a step function whose ordinate is y_f and a second function whose ordinates are those of y displaced downward by the value of y_f. Such a separation of y into its component functions is shown in figure 2. An analytical expression of the process may be derived as follows:

The transfer function of the system is

$$F(p) = \frac{\int_0^\infty ye^{-pt}\,dt}{\int_0^\infty xe^{-pt}\,dt} \tag{5}$$

Because of the linearity of the transformation

$$F(p) = \frac{\int_0^\infty (y-y_f)e^{-pt}\,dt + \int_0^\infty y_f e^{-pt}\,dt}{\int_0^\infty (x-x_f)e^{-pt}\,dt + \int_0^\infty x_f e^{-pt}\,dt}$$

$$= \frac{\int_0^\infty (y-y_f)e^{-pt}\,dt + \frac{y_f}{p}}{\int_0^\infty (x-x_f)e^{-pt}\,dt + \frac{x_f}{p}}$$

Setting $p = i\omega$ then gives

$$F(i\omega) = \frac{\int_0^\infty (y-y_f)e^{-i\omega t}\,dt + \frac{y_f}{i\omega}}{\int_0^\infty (x-x_f)e^{-i\omega t}\,dt + \frac{x_f}{i\omega}}$$

or

$$F(i\omega) = \frac{\int_0^\infty (y-y_f)\cos\omega t\,dt - i\left[\int_0^\infty (y-y_f)\sin\omega t\,dt + \frac{y_f}{\omega}\right]}{\int_0^\infty (x-x_f)\cos\omega t\,dt - i\left[\int_0^\infty (x-x_f)\sin\omega t\,dt + \frac{x_f}{\omega}\right]} \tag{13}$$

Equation (13) defines $F(i\omega)$ for all frequencies except $\omega = 0$ and $F(0)$ can readily be shown to be y_f/x_f.

Approximation of frequency-response function. - If the input and output functions are approximated by a series of straight-line segments, as shown in figure 3, then the integrals of equation (10) may be evaluated by expansion in finite sine and cosine series derived as follows: From figure 2, the integral $\int_0^\infty y'e^{-i\omega t}\,dt$ may be approximated by

$$\int_0^\infty y'e^{-i\omega t}\,dt = \sum_{k=0}^{r}\int_{t_k}^{t_{k+1}} y'_{k+1}\,e^{-i\omega t}\,dt$$

$$= \frac{1}{i\omega}\sum_{k=0}^{r} y'_{k+1}\left(e^{-i\omega t_k} - e^{-i\omega t_{k+1}}\right)$$

or

$$\int_0^\infty y'e^{-i\omega t}\,dt = \frac{1}{i\omega}\left(\sum_{k=0}^{r} y'_{k+1}\,e^{-i\omega t_k} - \sum_{k=0}^{r} y'_{k+1}\,e^{-i\omega t_{k+1}}\right) \tag{14}$$

If the first term of the first summation in equation (14) is evaluated and the index on the second summation is shifted to k-1, the expression becomes

$$\int_0^\infty y'e^{-i\omega t}\,dt = \frac{1}{i\omega}\left(y'_1 + \sum_{k=1}^{r} y'_{k+1}\,e^{-i\omega t_k} - \sum_{k=1}^{r} y'_k\,e^{-i\omega t_k}\right)$$

$$= \frac{1}{i\omega}\left[y'_1 - \sum_{k=1}^{r}\left(y'_k - y'_{k+1}\right)e^{-i\omega t_k}\right]$$

for $y'_{r+1} = 0$.

If the t's are evenly spaced,

$$t_{k+1} - t_k = t_1$$

Hence,

$$y'_k = \frac{y_k - y_{k-1}}{t_1}$$

or

$$y'_k - y'_{k+1} = \frac{2y_k - (y_{k-1} + y_{k+1})}{t_1}$$

Then

$$\int_0^\infty y' e^{-i\omega t}\, dt = \frac{1}{i\omega}\left[\frac{y_1}{t_1} - \sum_{k=1}^{r} \frac{2y_k - (y_{k-1} + y_{k+1})}{t_1} e^{-i\omega t_k}\right]$$

$$= \frac{-1}{\omega t_1}\left\{iy_1 - i\sum_{k=1}^{r}\left[2y_k - (y_{k-1} + y_{k+1})\right] e^{-i\omega t_k}\right\} \qquad (15)$$

Euler's relation,

$$e^{-i\omega t} = \cos \omega t - i \sin \omega t$$

applied to both members of equation (15), gives

$$\int_0^\infty y' \cos \omega t\, dt - i\int_0^\infty y' \sin \omega t\, dt =$$

$$\frac{-1}{\omega t_1}\left\{iy_1 - i\sum_{k=1}^{r}\left[2y_k - (y_{k-1} + y_{k+1})\right] \cos \omega t_k\right\} +$$

$$\frac{1}{\omega t_1}\sum_{k=1}^{r}\left[2y_k - (y_{k-1} + y_{k+1})\right] \sin \omega t_k$$

from which

$$\int_0^\infty y' \cos \omega t \, dt = \frac{1}{\omega t_1} \sum_{k=1}^{r} \left[2y_k - (y_{k-1} + y_{k+1}) \right] \sin \omega k t_1 \qquad (16)$$

and

$$\int_0^\infty y' \sin \omega t \, dt = \frac{1}{\omega t_1} \left\{ y_1 - \sum_{k=1}^{r} \left[2y_k - (y_{k-1} + y_{k+1}) \right] \cos \omega k t_1 \right\} \qquad (17)$$

where $kt_1 = t_k$.

A similar pair of expressions is obtained for the input function, with x merely replacing y throughout.

Oscillatory Final Conditions

The procedures thus far described apply only when the system comes to rest at some new equilibrium condition. If the output does not reach equilibrium but continues to oscillate about some mean value, a modification in the method is necessary in order to account for the oscillatory component.

System linearity permits resolution of the output y into two components y_1 and y_2, whose algebraic sum is the original output, as shown in figure 4. The block diagram equivalent to this resolution of y into components is given in figure 5. The dashed outline indicates the over-all transfer function of figure 1. From figure 5, $F(p)$ may be represented as the linear sum of two transfer functions, $F_1(p)$ and $F_2(p)$, operating in parallel.

The following equation may then be written:

$$F(p) = F_1(p) + F_2(p)$$

or

$$F(p) = \frac{L(y_1')}{L(x')} + \frac{L(y_2')}{L(x')}$$

$$= \frac{L(y_1') + p\,L(y_2)}{L(x')} \qquad (18)$$

If the steady oscillation is simple harmonic, the oscillatory component of the output may be expressed as

$$y_2 = A \sin (\beta t+\phi)$$

for $t > 0$. Then

$$p\, L(y_2) = \frac{Ap}{p^2+\beta^2} (\beta \cos \phi+p \sin \phi)$$

and

$$F(p) = \frac{\int_0^\infty y_1' e^{-pt}\, dt + \frac{Ap}{p^2+\beta^2} (\beta \cos \phi+p \sin \phi)}{\int_0^\infty x' e^{-pt}\, dt} \tag{19}$$

When p is replaced by $i\omega$ in equation (19), the frequency-response function becomes

$$F(i\omega) = \frac{\left(\int_0^\infty y_1' \cos \omega t\, dt - \frac{A\omega^2}{\beta^2-\omega^2} \sin \phi\right) - i\left(\int_0^\infty y_1' \sin \omega t\, dt - \frac{A\beta\omega}{\beta^2-\omega^2} \cos \phi\right)}{\int_0^\infty x' \cos \omega t\, dt - i \int_0^\infty x' \sin \omega t\, dt} \tag{20}$$

Treatment of Dead Time

Dead time in a linear system is the time difference between the initiation of a disturbance to the system and the beginning of a response to that disturbance. The mathematical equivalent of dead time then is a translation of the output function along the time axis in a positive direction; that is, if

$$L\left[y(t)\right] = F(p)\, L\left[x(t)\right] \tag{4}$$

for a system without dead time, then for the system with dead time

$$L\left[y(t-\Delta t)\right] = G(p)\, L\left[x(t)\right] \tag{21}$$

where $G(p)$ is different from $F(p)$. The relation between $G(p)$ and $F(p)$ is readily determined by an expansion in a Taylor's series of $y(t-\Delta t)$ in the neighborhood of t:

$$y(t-\Delta t) = y(t) - \Delta t y'(t) + \frac{\Delta t^2}{2!} y''(t) - \frac{\Delta t^3}{3!} y'''(t) + \ldots$$

Then

$$L\left[y(t-\Delta t)\right] = L\left[y(t)\right] - \Delta t\, L\left[y'(t)\right] + \frac{\Delta t^2}{2!} L\left[y''(t)\right] - \frac{\Delta t^3}{3!} L\left[y'''(t)\right] + \ldots$$

For initial conditions of equilibrium

$$L\left[y^n(t)\right] = p^n L\left[y(t)\right]$$

Hence,

$$L\left[y(t-\Delta t)\right] = L\left[y(t)\right] - \Delta tp\, L\left[y(t)\right] + \frac{\Delta t^2 p^2}{2!} L\left[y(t)\right] - \frac{\Delta t^3 p^3}{3!} L\left[y(t)\right] + \ldots$$

$$= \left(1 - \Delta tp + \frac{\Delta t^2 p^2}{2!} - \frac{\Delta t^3 p^3}{3!} + \ldots\right) L\left[y(t)\right]$$

$$= e^{-\Delta tp} L\left[y(t)\right] \tag{22}$$

From equations (21) and (22)

$$L\left[y(t)\right] = e^{\Delta tp} G(p)\, L\left[x(t)\right] \tag{23}$$

Comparison of equations (4) and (23) shows that

$$e^{\Delta tp} G(p) = F(p)$$

or

$$G(p) = e^{-\Delta tp} F(p)$$

from which the corresponding frequency-response functional relation is

$$G(i\omega) = e^{-i\Delta t\omega} F(i\omega) \tag{24}$$

It can easily be shown that the effect of dead time on the frequency-response function is to decrease the phase angle algebraically by the amount $\Delta t\omega$, leaving the amplitude ratio unchanged. In general, $F(i\omega)$ is a complex number and may be expressed as

$$F(i\omega) = Re^{i\theta}$$

From equation (24)

$$G(i\omega) = e^{-i\Delta t\omega}Re^{i\theta}$$

or

$$G(i\omega) = Re^{i(\theta-\Delta t\omega)}$$

EXAMPLES

Examples of the application of the method of this report to determination of the frequency-response function of several systems are found in figures 6 to 8. The use of the function thus found has not been covered because the detail with which such use has been treated elsewhere (for example, in references 1 and 2) makes extensive treatment in this report unwarranted.

Examples 1 and 2 illustrate the types of frequency response obtained from several different types of time function. A method of this report was used with the specified time functions to determine the points shown in figures 6 and 7. Frequency-response functions determined directly from the known transfer functions of the systems used were also plotted for comparison. Example 3 presents data for a turbine-propeller engine and shows the type of frequency response obtained, using a method of this report.

Example 1

A second-order system having the transfer function

$$F(p) = \frac{1}{p^2 + 2\zeta p + 1} \tag{25}$$

is shown in figure 6 for both the underdamped and critically damped cases. The frequency-response function of the system is obtained directly from equation (25) by replacing p by $i\omega$. For $\zeta = \frac{1}{2}$

$$F(i\omega) = \frac{1}{-\omega^2 + i\omega + 1}$$

$$= \frac{1 - \omega^2}{1 - \omega^2 + \omega^4} - i\,\frac{\omega}{1 - \omega^2 + \omega^4} \tag{26}$$

and for $\zeta = 1$

$$F(i\omega) = \frac{1}{-\omega^2 + 2i\omega + 1}$$

$$= \frac{1 - \omega^2}{(1+\omega^2)^2} - i\,\frac{2\omega}{(1+\omega^2)^2} \tag{27}$$

System response to a unit step input is given for $\zeta = \frac{1}{2}$ by

$$y = 1 - \frac{2\sqrt{3}}{3} e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t + \frac{\pi}{3}\right) \tag{28}$$

and for $\zeta = 1$ by

$$y = 1 - e^{-t}(1 + t) \tag{29}$$

Frequency-response curves calculated from equations (26) and (27) are shown in figures 6(c) and 6(d). The rolling-sphere analyzer was used on the transient-response curves determined from equations (28) and (29) to obtain from equation (10) the points shown on the frequency-response curves of figure 6. Because a unit step input was used, the denominator of the right-hand side of equation (10) is equal to 1 for all values of frequency and $F(i\omega)$ can be obtained by operation of the analyzer over the output curves only.

Example 2

A first-order system is shown in figure 7. If the system has no dead time, the transfer function is

$$F(p) = \frac{1}{1 + \tau p}$$

and the corresponding frequency-response function is

$$F(i\omega) = \frac{1}{1 + \tau i\omega}$$

$$= \frac{1}{1 + \tau^2\omega^2} - i\frac{\tau\omega}{1 + \tau^2\omega^2} \qquad (31)$$

For a unit step input the output is

$$y = 1 - e^{-\frac{t}{\tau}} \qquad (32)$$

If the system has a dead time Δt, the transfer function becomes

$$G(p) = \frac{e^{-\Delta tp}}{1 + \tau p} \qquad (33)$$

and

$$G(i\omega) = e^{-i\Delta t\omega} F(i\omega) \qquad (34)$$

The response to a unit step input then is

$$y = 1 - e^{-\frac{t-\Delta t}{\tau}} \qquad (35)$$

for $t \geq \Delta t$. The frequency-response curves in figures 7(d) and 7(e) were calculated from equations (31) and (34). The function $F(i\omega)$ was found from equation (31) and $G(i\omega)$ was then determined from $F(i\omega)$ by increasing the phase angle of $F(i\omega)$ negatively by the amount $\Delta t\omega$, leaving $|F(i\omega)|$ unchanged. The rolling-sphere analyzer was used on the output curves of equations (32) and (35) to obtain from equation (10) the points shown on the frequency-response curves of figure 7. The use of a unit step input to the system makes the denominator of the right-hand side of equation (10) equal to 1 for all values of frequency. The frequency-response function can therefore be obtained by operation of the analyzer over the output curves only.

Example 3

Experimental transient data obtained at this laboratory for the speed response of a turbine-propeller engine to changes in propeller-blade angle are shown in figure 8(a). Because the frequency-response function is expressed as a ratio, units of the variables involved are unimportant and the ordinate of the data curves has been calibrated in relative values, the difference between initial and final values of the curves being set equal to unity.

The frequency-response function $F(i\omega)$ of this engine for response of engine speed to changes in propeller-blade angle was found from equation (10). The rolling-sphere analyzer was used on the data. In figure 8(b), $F(i\omega)$ is shown as a frequency-locus plot in the complex plane; figure 8(c) is a plot of amplitude ratio and phase angle against frequency. The closeness with which the data approach a semicircle in the complex plane indicates that the response of the system is substantially first-order. This inference is further borne out by the amplitude-ratio plot. The asymptotes and the solid curve were calculated for an exact first-order system.

SUMMARY OF RESULTS

Methods are presented for the determination of the frequency-response function of any linear system from general correlative time-response data for input and output of that system. Equations were developed for system equilibrium and oscillatory final conditions and for the effect of system dead time on the frequency-response function.

For illustrative purposes, a method of this report was applied to several linear systems for which the frequency-response functions were known.

Experimental data obtained at this laboratory for the speed response of a turbine-propeller engine to change in propeller-blade angle are presented and analyzed.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, April 11, 1949.

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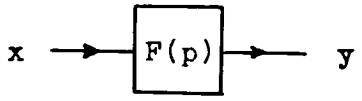
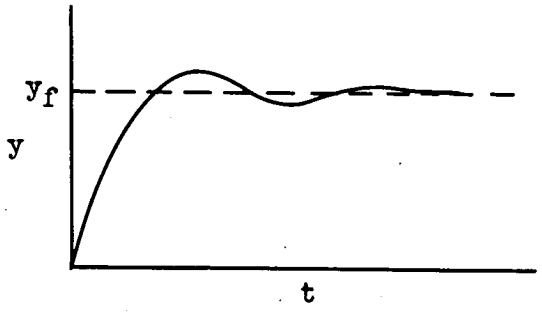
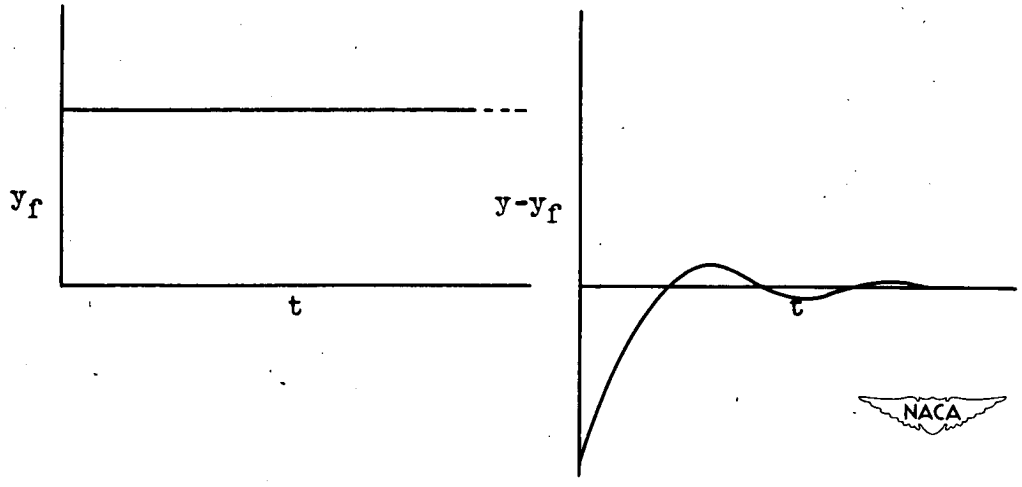


Figure 1. - Block diagram of linear system.



(a) System-output time function.



(b) Components of system-output time function.

Figure 2. - Separation of system-output time function into components for equilibrium final conditions.

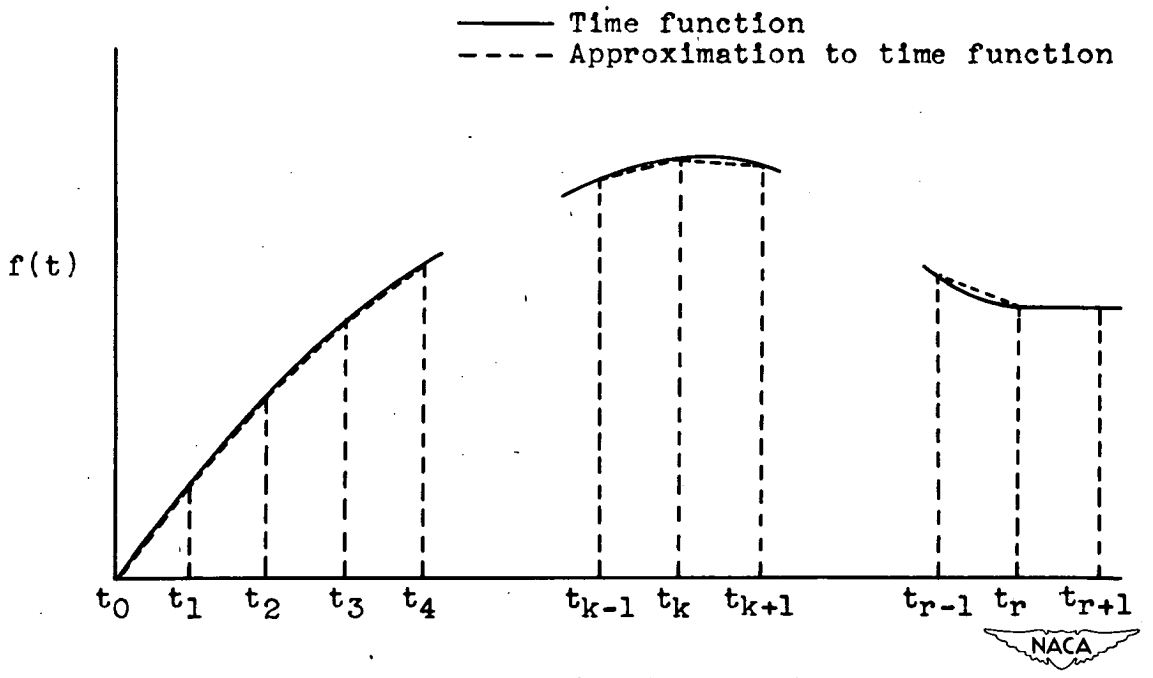
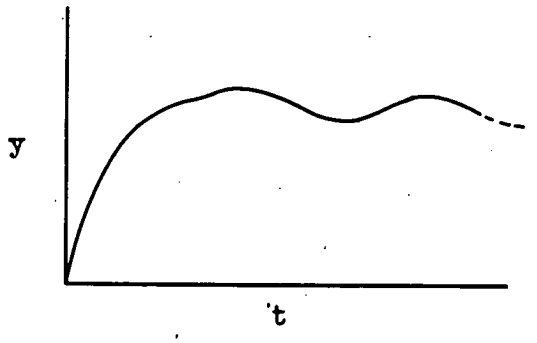
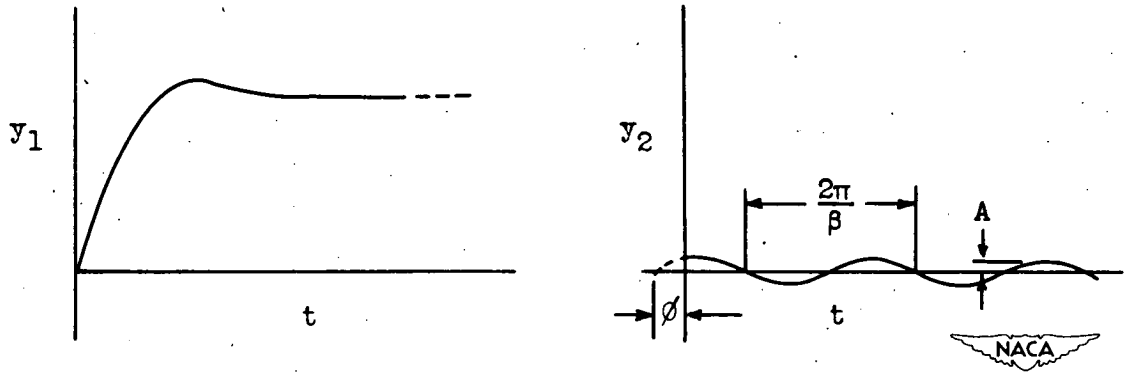


Figure 3. - Approximation to time function.



(a) System-output time function.



(b) Components of system-output time function.

Figure 4. - Separation of system-output time function into components for oscillatory final conditions.

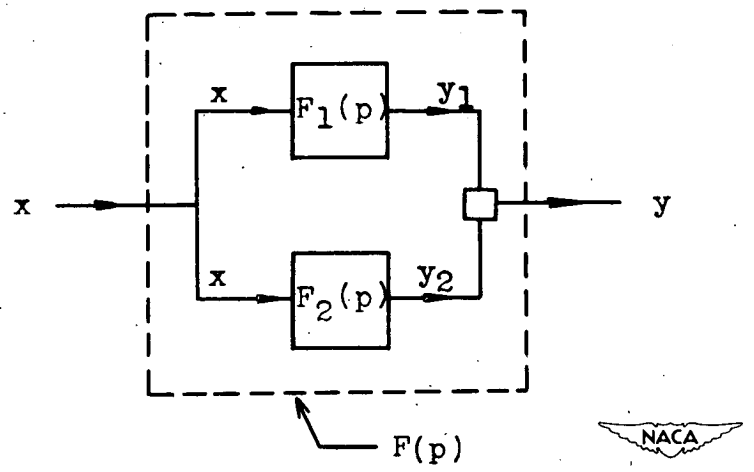
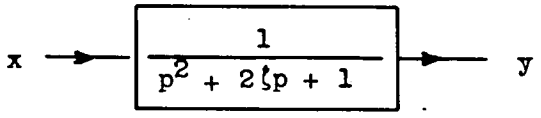
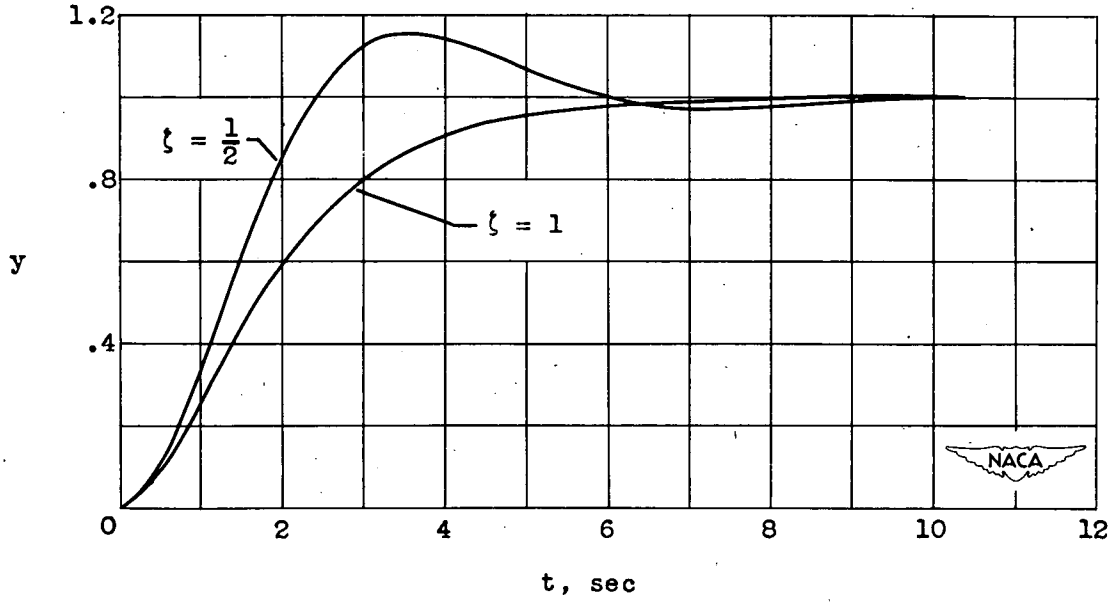


Figure 5. - Block-diagram representation of linear system for oscillatory final conditions.

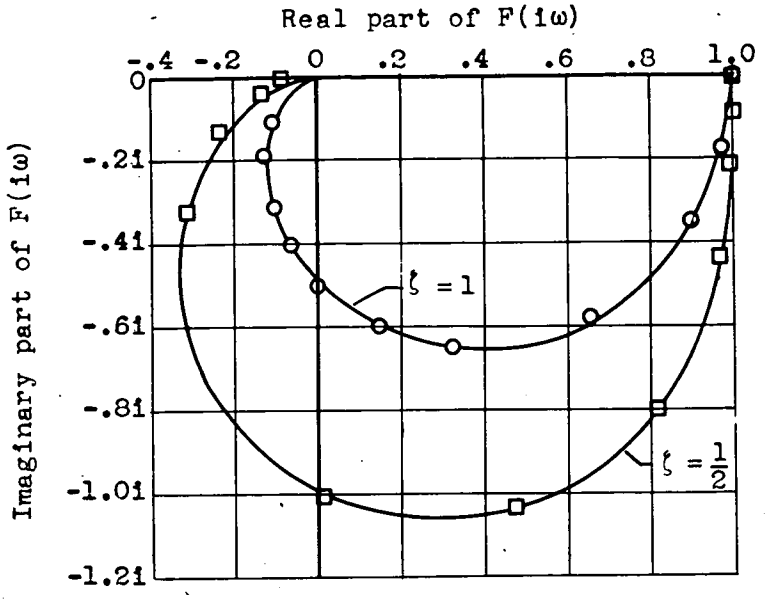


(a) Second-order system representation.

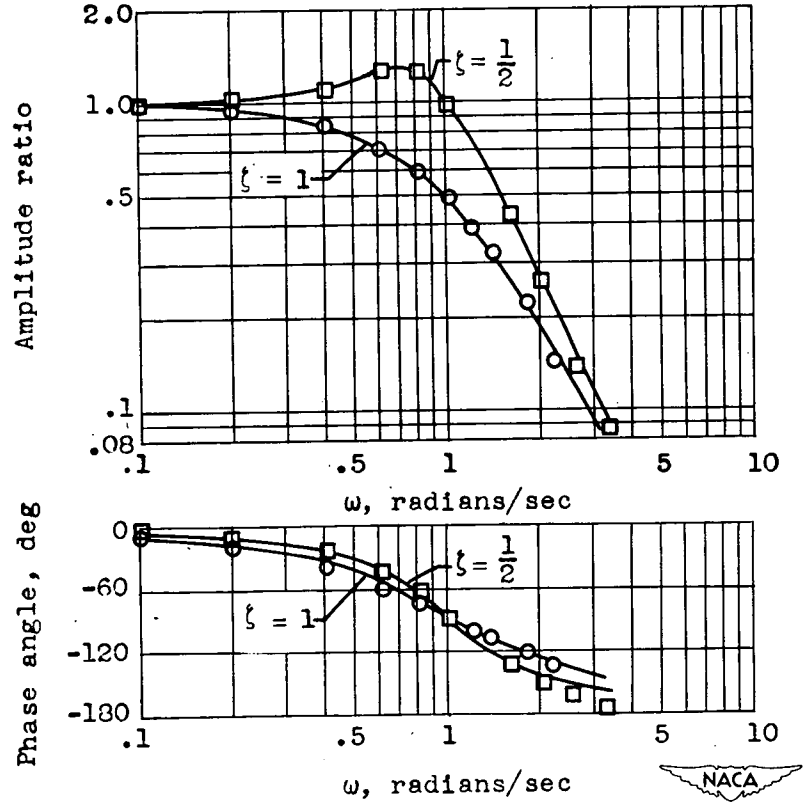


(b) Response of system to unit step input.

Figure 6. - Frequency-response function for second-order system.

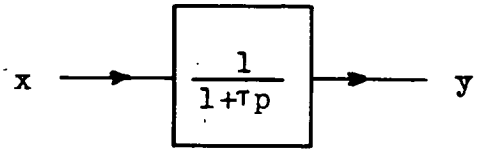


(c) Frequency-response function in complex plane.

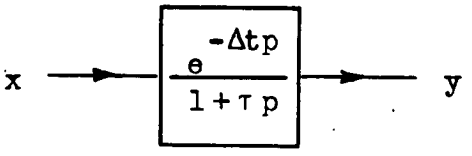


(d) Amplitude ratio and phase angle of frequency-response function against ω.

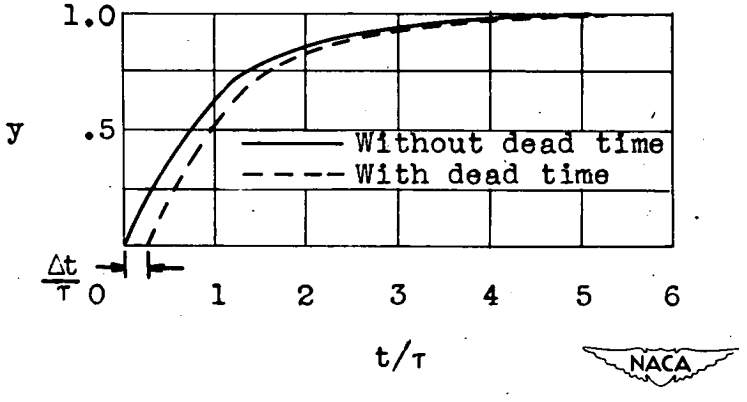
Figure 6. - Concluded. Frequency-response function for second-order system.



(a) Representation of exact first-order system with no dead time.

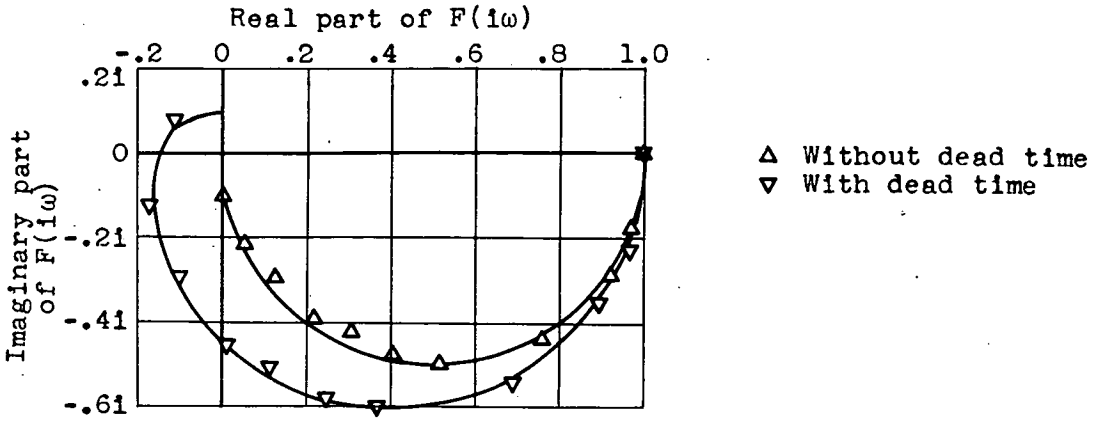


(b) Representation of exact first-order system with dead time.

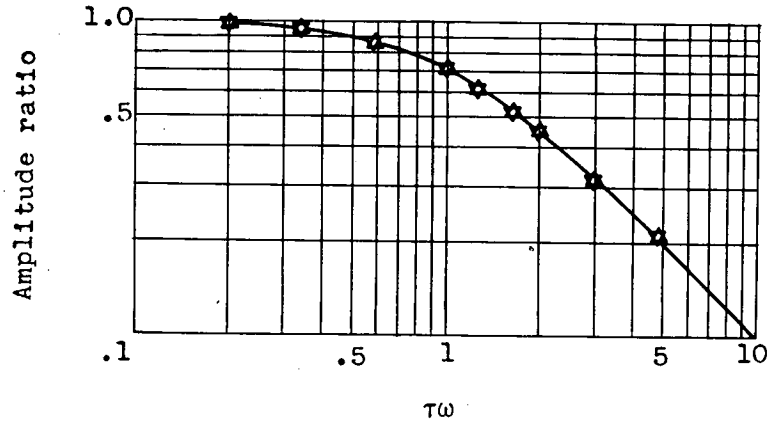


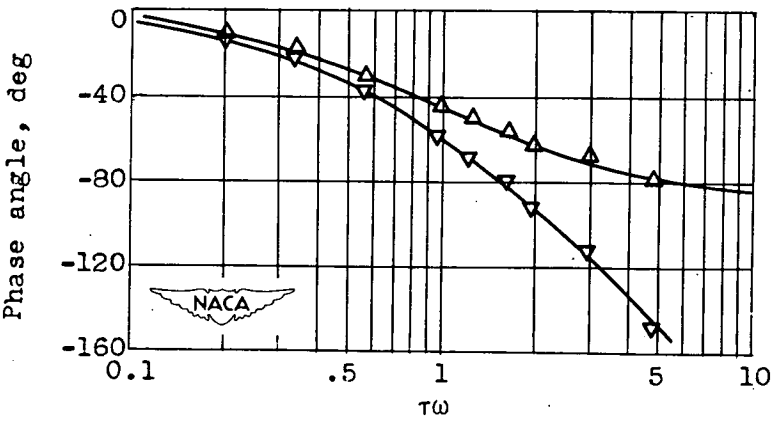
(c) System response to unit step input.

Figure 7. - Frequency-response function for first-order system.



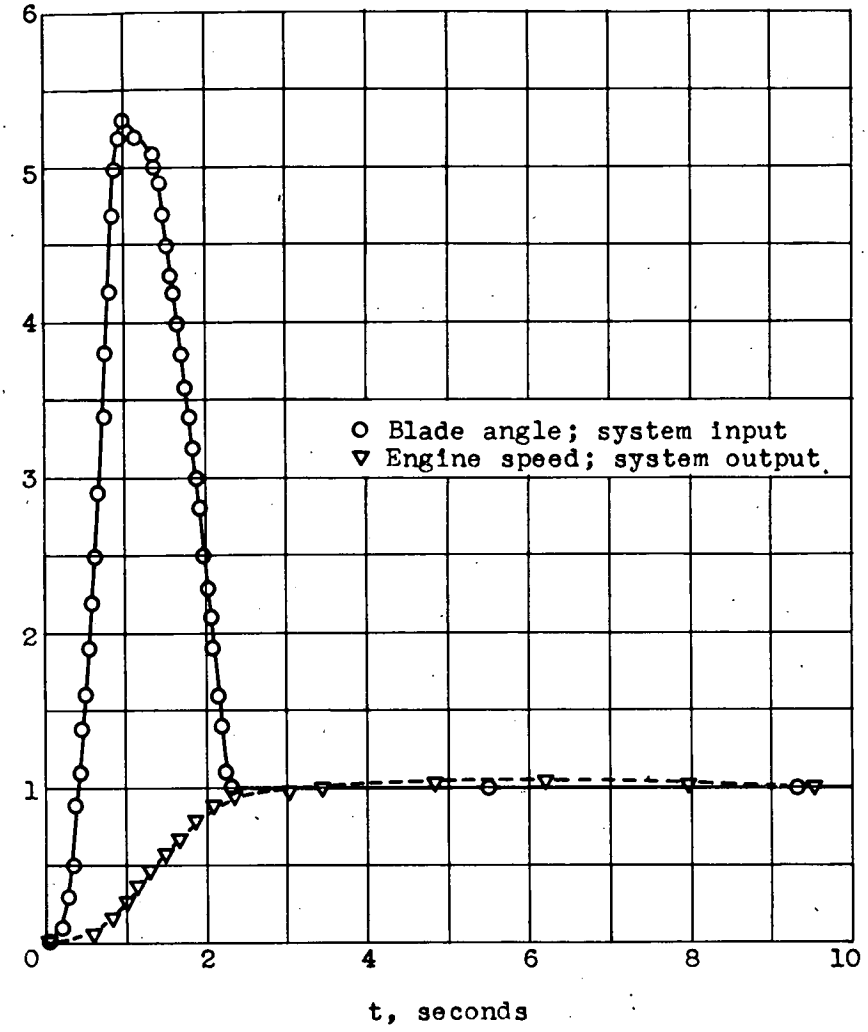
(d) Frequency-response function in complex plane.



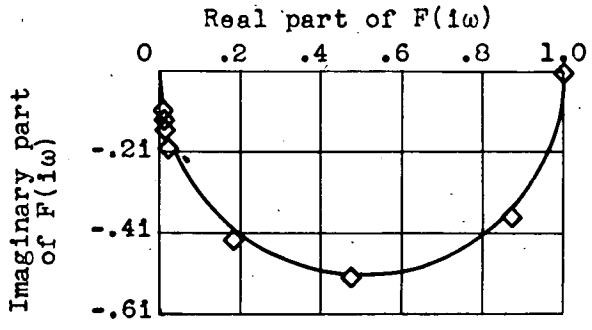


(e) Amplitude ratio and phase angle of frequency-response function against $\tau\omega$.

Figure 7. - Concluded. Frequency-response function for first-order system.

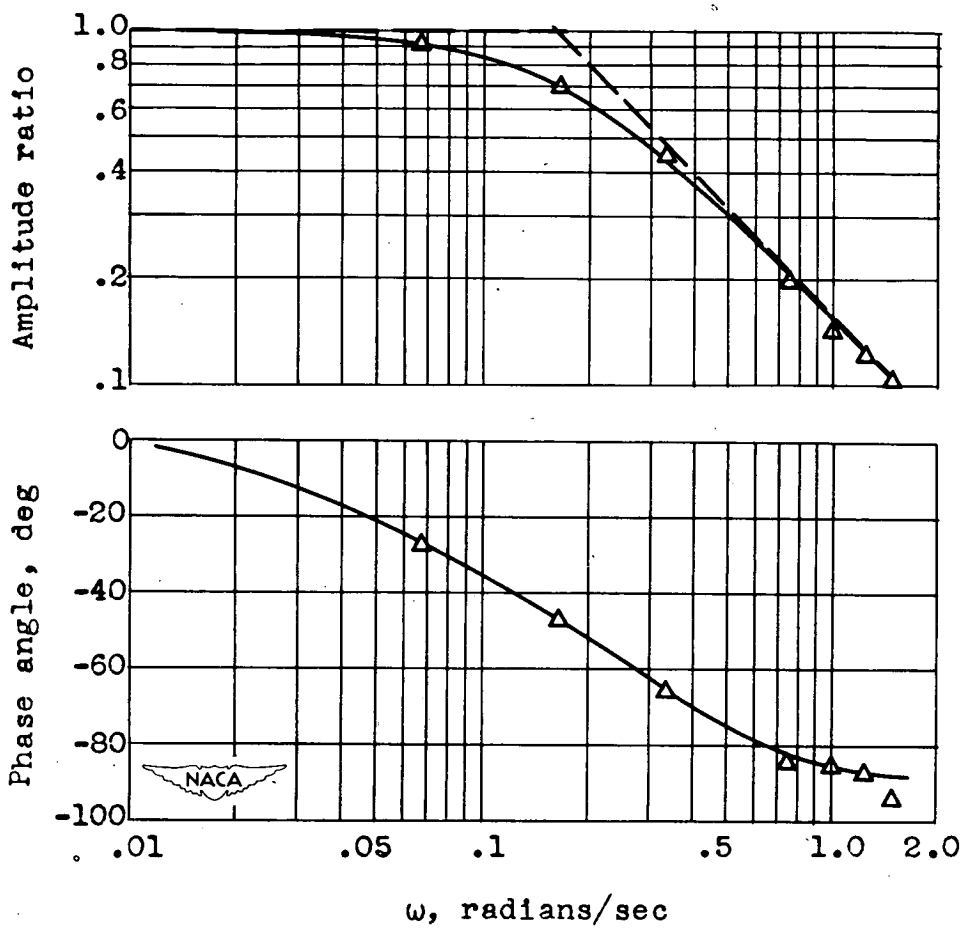


(a) Input and output data for turbine-propeller engine.



(b) Frequency-response function of turbine-propeller engine plotted in complex plane.

Figure 8. - Frequency-response function of turbine-propeller engine.



(c) Amplitude ratio and phase angle of turbine-propeller frequency-response function against ω.

Figure 8. - Concluded. Frequency-response function of turbine-propeller engine.